Practice Second Midterm Exam II

This exam is closed-book and closed-computer. You may have a double-sided, 8.5" × 11" sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices (laptops, cell phones, etc.) during the course of this exam. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

> On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 36 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to prepare for the exam. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

Question	Points
(1) Binary Relations	/ 8
(2) Functions	/ 10
(3) Induction	/ 10
(4) The Pigeonhole Principle	/ 8
	/ 36

Problem One: Binary Relations   (8 Points)

(Midterm Exam, Winter 2016)

On Problem Set Three, you were given several examples of concrete relations and functions and then asked to prove properties about those objects. In this problem, we're going to define a new binary relation over the set $\mathbb{R}$, then ask you to prove an important property about that new relation. In the course of doing so, you'll be able to demonstrate what you've learned about writing proofs that refer back to core terms and definitions.

Let $S \subseteq \mathbb{R}$ be an arbitrary set of real numbers with the following properties:

- S contains zero (that is, $0 \in S$).

- S is ***closed under addition*** (that is, if $x \in S$ and $y \in S$, then $x+y \in S$).

- S is ***closed under additive inverses*** (that is, if $x \in S$, then $-x \in S$).

Consider the binary relation ~ over $\mathbb{R}$ defined as follows:

$$x \sim y \quad \text{if} \quad y - x \in S.$$

Prove that ~ is an equivalence relation over $\mathbb{R}$.

(Extra space for your answer to Problem One, if you need it.)

Problem Two: Functions   (10 Points)

(A bespoke problem made just for this practice exam!)

As a refresher, two functions $r : A \to B$ and $s : A \to B$ are equal to one another (denoted $r = s$) if the following is true:

$$\forall x \in A.\ r(x) = s(x).$$

In other words, two functions are equal if they have the same domain and codomain and produce the same outputs on all inputs.

Let's introduce a new definition. A function $h : A \to A$ is called an ***involution*** if, for any $x \in A$, we have that

$$(h \circ h)(x)\ =\ x.$$

Prove that if $f : A \to A$ and $g : A \to A$ are involutions, then the function $g \circ f$ is an involution if and only if $g \circ f = f \circ g$. You may want to use the fact that function composition is ***associative***: for any functions r, s, and t with the appropriate domains and codomains, we have

$$(r \circ s) \circ t\ =\ r \circ (s \circ t)$$

(Extra space for your answer to Problem Two, if you need it)

Problem Three: Induction   (10 Points)

(An old CS103 question from Way Back In The Day)

A ***directed tree*** is a structure used to represent hierarchical information. A directed tree consists of a set of ***nodes***. Each node can have zero or more ***child nodes***. Every node has exactly one parent except for a special node called the ***root node***, which has no parent.

Binomial trees are a specific family of directed trees defined as follows: a binomial tree of order n is a single node with n children, which are binomial trees of order $0, 1, 2, \ldots, n - 1$. For example, here are pictures of binomial trees of orders 0, 1, 2, and 3:
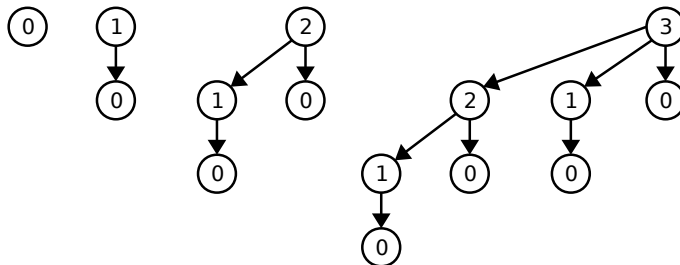

Prove by induction that a binomial tree of order n has exactly 2^n nodes. Your proof should use to the formal definition of binomial trees given above – if you want to use any other facts about binomial trees in your proof of the main result, you will need to prove those other facts first.

(Extra space for your answer to Problem Three, if you need it.)

Problem Four: The Pigeonhole Principle   (8 Points)

(CS103 Midterm, Fall 2012)

Suppose that you color every point in the Cartesian plane one of four colors (say, red, green, blue, and yellow). Prove that no matter how you color the plane, there will always be a trapezoid whose corners are all the same color. Recall that a trapezoid is a quadrilateral with at least two parallel sides. For example, all of the following figures are trapezoids:
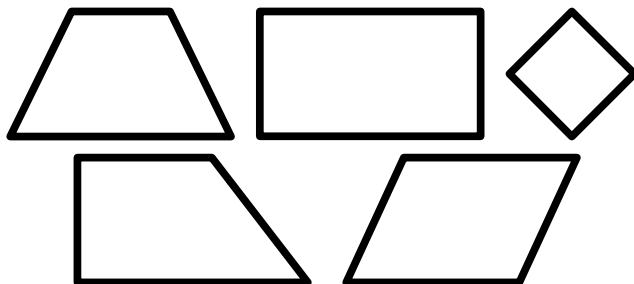
As a hint, consider dropping a 5×5 grid of points into the plane. Each of those points will be assigned one of four colors. Can you form a trapezoid from some of those points?